

# Nonabelian Kaluza-Klein Dyon

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November 3, 2009

## Abstract

We propose a new nonabelian Kaluza-Klein dyon solution in the seven-dimensional spacetimes. This is a Wu-Yang-like Kaluza-Klein dyon. The dyonic metric is spherically symmetric, two spherical coordinate systems are used, and the metric admits  $SO(3)$  Killing vectors. It can be verified that the Einstein equation is satisfied in seven-dimensional spacetimes. The stress-energy tensor of the Yang-Mills field in Einstein equation is derived from Ricci tensor automatically, not put by hand from outside. The Ricci scalar curvature  $\bar{R}$  vanishes. The four-dimensional part of the dyonic metric is just the Schwarzschild black hole metric.

PACS numbers: 04.50.Cd

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# 1 Introduction

The well-known Gross-Perry-Sorkin solution [1] [2] in five-dimensional Kaluza-Klein (KK) theory is the simplest KK magnetic monopole solution associated with  $U(1)$  abelian group. It has long been puzzled that whether the nonabelian KK Wu-Yang-like [3] monopole [4] or dyon solutions may exist? The Wu-Yang monopole means that it is a spherically symmetric point-like monopole without Dirac string. In this paper we will present a nonabelian Wu-Yang-like KK dyon solution in the seven-dimensional spacetimes. By calculating the Christoffel symbols and the Ricci tensor, it can be shown that the Einstein equation is satisfied. In the next paper, we will show an alternative method by using the orthonormal frame and Cartan's structure equations. The Ricci tensor in that paper can be obtained via the affine spin connection one-form. The results from these two different methods are coincident.

Suppose the line element of the Kaluza-Klein theory [5] in  $(4 + N)$  dimensions can be written as

$$d\bar{s}^2 = \bar{g}_{AB}d\bar{x}^A d\bar{x}^B \quad (1)$$

$$= g_{\mu\nu}(x)dx^\mu dx^\nu + \gamma_{mn}(y)(dy^m + B_\mu^m dx^\mu)(dy^n + B_\nu^n dx^\nu), \quad (2)$$

where  $x$  parametrizes four-dimensional spacetimes,  $y$  parametrizes extra dimensions. We use  $A, B, C...$  indices to represent the total spacetimes;  $\mu, \nu, \rho...$  to represent the four-dimensional spacetimes;  $m, n, l...$  to represent the extra dimensions.  $g_{\mu\nu}$  is only a function of  $x$ , and  $\gamma_{mn}$  is only a function of  $y$ . The metric tensor is

$$\bar{g}_{AB} = \begin{pmatrix} \bar{g}_{\mu\nu} & \bar{g}_{\mu n} \\ \bar{g}_{m\nu} & \bar{g}_{mn} \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} g_{\mu\nu} + \gamma_{mn}B_\mu^m B_\nu^n & B_\mu^m \gamma_{mn} \\ \gamma_{mn}B_\nu^n & \gamma_{mn} \end{pmatrix}, \quad (4)$$

while the inverse metric tensor is

$$\bar{g}^{AB} = \begin{pmatrix} \bar{g}^{\mu\nu} & \bar{g}^{\mu n} \\ \bar{g}^{m\nu} & \bar{g}^{mn} \end{pmatrix} \quad (5)$$

$$= \begin{pmatrix} g^{\mu\nu} & -g^{\mu\nu} B_\nu^n \\ -B_\lambda^m g^{\lambda\nu} & \gamma^{mn} + B_\lambda^m B_\sigma^n g^{\lambda\sigma} \end{pmatrix}. \quad (6)$$

$B_\mu^m$  cannot be identified as the Yang-Mills field. To extract the true Yang-Mills field, one has to introduce the Killing vectors

$$L_a \equiv -i\zeta_a^m \partial_m, \quad (7)$$

which generate a Lie algebra,

$$[L_a, L_b] = if_{ab}^c L_c, \quad (8)$$

associated with some symmetry.  $a, b, c...$  are the gauge group indices.  $f_{ab}^c$  are the structure constants of a Lie algebra. There is no need to differentiate between the

lower and upper indices for the gauge groups. Inserting  $L_a$  of (7) into equation (8), one gets the Killing's equation

$$\zeta_a^m \partial_m \zeta_b^n - \zeta_b^m \partial_m \zeta_a^n = -f_{ab}^c \zeta_c^n. \quad (9)$$

With these Killing vectors, one can define

$$B_\mu^m = \zeta_a^m A_\mu^a, \quad (10)$$

where  $A_\mu^a$  is the true Yang-Mills field and  $\zeta_a^m$  is only a function of  $y$ .

The Christoffel symbols defined by the form,

$$\bar{\Gamma}_{AB}^C = \frac{1}{2} \bar{g}^{CD} (\partial_A \bar{g}_{BD} + \partial_B \bar{g}_{AD} - \partial_D \bar{g}_{AB}), \quad (11)$$

can be given more explicitly by

$$\begin{aligned} \bar{\Gamma}_{\mu\nu}^\alpha &= \Gamma_{\mu\nu}^\alpha + \frac{1}{2} g^{\alpha\lambda} \gamma_{mn} (B_\nu^m \tilde{\mathcal{F}}_{\mu\lambda}^n + B_\mu^m \tilde{\mathcal{F}}_{\nu\lambda}^n) + \frac{1}{2} g^{\alpha\lambda} \gamma_{mn} (B_\mu^m B_\nu^l + B_\nu^m B_\mu^l) \partial_l B_\lambda^n \\ &\quad + \frac{1}{2} g^{\alpha\lambda} B_\mu^m B_\nu^n B_\lambda^l \partial_l \gamma_{mn}, \end{aligned} \quad (12)$$

$$\bar{\Gamma}_{\mu\nu}^m = -B_\lambda^m \bar{\Gamma}_{\mu\nu}^\lambda + \frac{1}{2} (\partial_\mu B_\nu^m + \partial_\nu B_\mu^m) - \frac{1}{2} \gamma^{mn} \partial_n (B_\mu^s B_\nu^t \gamma_{st}), \quad (13)$$

$$\bar{\Gamma}_{m\nu}^\alpha = \frac{1}{2} g^{\alpha\lambda} \gamma_{ms} \tilde{\mathcal{F}}_{\nu\lambda}^s + \frac{1}{2} g^{\alpha\lambda} B_\lambda^s B_\nu^l (\partial_l \gamma_{ms}) + \frac{1}{2} g^{\alpha\lambda} B_\nu^l (\gamma_{ms} \partial_l B_\lambda^s + \gamma_{ls} \partial_m B_\lambda^s), \quad (14)$$

$$\bar{\Gamma}_{mn}^\mu = \frac{1}{2} g^{\mu\nu} \gamma_{nl} \partial_m B_\nu^l + \frac{1}{2} g^{\mu\nu} \gamma_{ml} \partial_n B_\nu^l + \frac{1}{2} g^{\mu\nu} B_\nu^l \partial_l \gamma_{mn}, \quad (15)$$

$$\bar{\Gamma}_{n\nu}^m = -B_\lambda^m \bar{\Gamma}_{n\nu}^\lambda + \frac{1}{2} \gamma^{ml} \partial_n (B_\nu^s \gamma_{sl}) - \frac{1}{2} \gamma^{ml} \partial_l (B_\nu^s \gamma_{sn}), \quad (16)$$

$$\bar{\Gamma}_{nl}^m = \tilde{\Gamma}_{nl}^m - \frac{1}{2} B_\lambda^m g^{\lambda\mu} (\gamma_{ls} \partial_n B_\mu^s + \gamma_{ns} \partial_l B_\mu^s) - \frac{1}{2} B_\lambda^m g^{\lambda\mu} B_\mu^s \partial_s \gamma_{nl}, \quad (17)$$

where

$$\tilde{\mathcal{F}}_{\mu\nu}^m \equiv \partial_\mu B_\nu^m - \partial_\nu B_\mu^m - (\mu \leftrightarrow \nu) \quad (18)$$

$$= \zeta_a^m F_{\mu\nu}^a. \quad (19)$$

$F_{\mu\nu}^a$  is the true field strength tensor of the Yang-Mills field,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}^a A_\mu^b A_\nu^c. \quad (20)$$

$\Gamma_{\mu\nu}^\alpha$  in (12) are the Christoffel symbols of the four-dimensional spacetimes,  $\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ , while  $\tilde{\Gamma}_{nl}^m$  in (17) are the Christoffel symbols of the extra dimensions,  $\tilde{\Gamma}_{nl}^m = \frac{1}{2} \gamma^{ms} (\partial_n \gamma_{ls} + \partial_l \gamma_{ns} - \partial_s \gamma_{nl})$ .

## 2 SO(3) Kaluza-Klein Dyon

Now let us consider an ansatz of the Kaluza-Klein dyonic metric admitting  $SO(3)$  Killing vectors. The metric is spherically symmetric, not only for the four-dimensional spacetimes but also for the extra dimensions. The line element can be written as the form

$$\begin{aligned} d\bar{s}^2 = & - e^{2\Psi} dt^2 \\ & + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ & + e^{2\chi} (dR + B_\mu^5 dx^\mu)^2 + R^2 (d\Theta + B_\mu^6 dx^\mu)^2 + R^2 \sin^2 \Theta (d\Phi + B_\mu^7 dx^\mu)^2. \end{aligned} \quad (21)$$

$r, \theta, \phi$  are three coordinates of the ordinary three-dimensional spherical coordinate system,  $(r, \theta, \phi) = (\bar{x}^1, \bar{x}^2, \bar{x}^3) = (x^1, x^2, x^3)$ .  $R, \Theta, \Phi$  are another three coordinates of the spherical coordinate system in the extra dimensions,  $(R, \Theta, \Phi) = (\bar{x}^5, \bar{x}^6, \bar{x}^7) = (y^5, y^6, y^7)$ .  $\Psi$  and  $\Lambda$  are two functions of  $r$ , while  $\chi$  is only a function of  $R$ . We have  $g_{00} = -e^{2\Psi}$ ,  $g_{11} = e^{2\Lambda}$ ,  $g_{22} = r^2$ ,  $g_{33} = r^2 \sin^2 \theta$ ,  $\gamma_{55} = e^{2\chi}$ ,  $\gamma_{66} = R^2$ ,  $\gamma_{77} = R^2 \sin^2 \Theta$ .

The gauge-field components of the Wu-Yang-like KK dyon are

$$A_0^a = \frac{1}{r} \hat{r}^a, \quad A_1^a = 0, \quad (22)$$

$$A_2^a = -\hat{\phi}^a, \quad A_3^a = \sin \theta \hat{\theta}^a. \quad (23)$$

$A_1^a, A_2^a, A_3^a$  are just the spherical coordinate representation of the Wu-Yang monopole field in the ordinary gauge theory of four-dimensional spacetimes. The electric field of the KK dyon is

$$F_{01}^a = \frac{1}{r^2} \hat{r}^a, \quad (24)$$

while the magnetic field is

$$F_{23}^a = -\sin \theta \hat{r}^a. \quad (25)$$

Since the relevant Killing vectors are

$$L_1 = -i \left( -\sin \Phi \frac{\partial}{\partial \Theta} - \cot \Theta \cos \Phi \frac{\partial}{\partial \Phi} \right), \quad (26)$$

$$L_2 = -i \left( \cos \Phi \frac{\partial}{\partial \Theta} - \cot \Theta \sin \Phi \frac{\partial}{\partial \Phi} \right), \quad (27)$$

$$L_3 = -i \left( \frac{\partial}{\partial \Phi} \right), \quad (28)$$

which are just the generators of the  $SO(3)$  group, then one has

$$\zeta_a^5 = 0, \quad (29)$$

$$\zeta_a^6 = \hat{\Phi}_a, \quad (30)$$

$$\zeta_a^7 = -\frac{1}{\sin \Theta} \hat{\Theta}_a. \quad (31)$$

$\hat{R}_a, \hat{\Theta}_a, \hat{\Phi}_a$  are three unit vectors of the spherical coordinate system of the extra dimensions. It can be checked that above three equations of  $\zeta_a^m$  also satisfy the Killing equation (9). The fields  $B_\mu^m$  in (10) can be rewritten as

$$B_\mu^5 = 0, \quad B_1^m = 0, \quad (32)$$

$$B_0^6 = \frac{1}{r} \hat{r} \cdot \hat{\Phi}, \quad B_0^7 = -\frac{1}{r \sin \Theta} \hat{r} \cdot \hat{\Theta}, \quad (33)$$

$$B_2^6 = -\hat{\phi} \cdot \hat{\Phi}, \quad B_2^7 = \frac{1}{\sin \Theta} \hat{\phi} \cdot \hat{\Theta}, \quad (34)$$

$$B_3^6 = \sin \theta \hat{\theta} \cdot \hat{\Phi}, \quad B_3^7 = -\frac{\sin \theta}{\sin \Theta} \hat{\theta} \cdot \hat{\Theta}. \quad (35)$$

The nonzero components of  $\tilde{\mathcal{F}}_{\mu\nu}^m$  are

$$\tilde{\mathcal{F}}_{01}^6 = \frac{1}{r^2} \hat{r} \cdot \hat{\Phi}, \quad \tilde{\mathcal{F}}_{01}^7 = \frac{-1}{r^2 \sin \Theta} \hat{r} \cdot \hat{\Theta}, \quad (36)$$

$$\tilde{\mathcal{F}}_{23}^6 = -\sin \theta \hat{r} \cdot \hat{\Phi}, \quad \tilde{\mathcal{F}}_{23}^7 = \frac{\sin \theta}{\sin \Theta} \hat{r} \cdot \hat{\Theta}. \quad (37)$$

Because of the chosen dyonic metric, there exist the following identities:

$$\gamma_{77} \partial_6 B_\mu^7 + \gamma_{66} \partial_7 B_\mu^6 = 0, \quad (38)$$

$$\gamma_{77} \partial_7 B_\mu^7 + \frac{1}{2} B_\mu^6 \partial_6 \gamma_{77} = 0, \quad (39)$$

which simplify our calculations drastically. For example, from (12) and (14), we get the simplified Christoffel symbols,

$$\bar{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + \frac{1}{2} g^{\alpha\lambda} \gamma_{mn} (B_\nu^m \tilde{\mathcal{F}}_{\mu\lambda}^n + B_\mu^m \tilde{\mathcal{F}}_{\nu\lambda}^n), \quad (40)$$

$$\bar{\Gamma}_{m\nu}^\alpha = \frac{1}{2} g^{\alpha\lambda} \gamma_{ms} \tilde{\mathcal{F}}_{\nu\lambda}^s. \quad (41)$$

Among 196 ( $= 7 \times 28$ ) independent components of the Christoffel symbols, there are 82 independent components are nonzero. Substituting these into the Ricci tensor,

$$\bar{R}_{BD} = \partial_A \bar{\Gamma}_{BD}^A - \partial_D \bar{\Gamma}_{BA}^A + \bar{\Gamma}_{AE}^A \bar{\Gamma}_{BD}^E - \bar{\Gamma}_{DE}^A \bar{\Gamma}_{AB}^E. \quad (42)$$

During the lengthy calculations, one will use frequently the following identities:

$$\partial_\theta \hat{r} = \hat{\theta}, \quad \partial_\Theta \hat{R} = \hat{\Theta}, \quad (43)$$

$$\partial_\theta \hat{\theta} = -\hat{r}, \quad \partial_\Theta \hat{\Theta} = -\hat{R}, \quad (44)$$

$$\partial_\phi \hat{r} = \sin \theta \hat{\phi}, \quad \partial_\Phi \hat{R} = \sin \Theta \hat{\Phi}, \quad (45)$$

$$\partial_\phi \hat{\theta} = \cos \theta \hat{\phi}, \quad \partial_\Phi \hat{\Theta} = \cos \Theta \hat{\Phi}, \quad (46)$$

$$\partial_\phi \hat{\phi} = -\sin \theta \hat{r} - \cos \theta \hat{\theta}, \quad (47)$$

$$\partial_\Phi \hat{\Phi} = -\sin \Theta \hat{R} - \cos \Theta \hat{\Theta}, \quad (48)$$

$$\hat{r}_a \hat{r}_b + \hat{\theta}_a \hat{\theta}_b + \hat{\phi}_a \hat{\phi}_b = \delta_{ab}, \quad (49)$$

$$\hat{R}_a \hat{R}_b + \hat{\Theta}_a \hat{\Theta}_b + \hat{\Phi}_a \hat{\Phi}_b = \delta_{ab}. \quad (50)$$

$$(\hat{\theta} \cdot \hat{\Theta}) (\hat{\phi} \cdot \hat{\Phi}) - (\hat{\theta} \cdot \hat{\Phi}) (\hat{\phi} \cdot \hat{\Theta}) = \hat{r} \cdot \hat{R} \quad (51)$$

$$(\hat{r} \cdot \hat{\Theta}) (\hat{\theta} \cdot \hat{\Phi}) - (\hat{r} \cdot \hat{\Phi}) (\hat{\theta} \cdot \hat{\Theta}) = \hat{\phi} \cdot \hat{R} \quad (52)$$

After many miracle cancellations, it can be found that the components of the dyonic metric are

$$g_{00} = -(1 - \frac{r_s}{r}), \quad (53)$$

$$g_{11} = (1 - \frac{r_s}{r})^{-1}, \quad (54)$$

$$\gamma_{55} = 1, \quad (55)$$

where  $r_s$  is the Schwarzschild radius. Then  $\bar{R}_{mn}$ ,  $\bar{R}_{\mu m}$  and almost  $\bar{R}_{\mu\nu}$  are equal to zero except

$$\bar{R}_{00} = -\frac{1}{2} g^{11} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}, \quad (56)$$

$$\bar{R}_{11} = -\frac{1}{2} g^{00} \frac{R^2}{r^4} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}, \quad (57)$$

$$\bar{R}_{22} = -\frac{R^2}{2r^2} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}, \quad (58)$$

$$\bar{R}_{33} = -\frac{R^2 \sin^2 \theta}{2r^2} \{(\hat{r} \cdot \hat{\Theta})^2 + (\hat{r} \cdot \hat{\Phi})^2\}. \quad (59)$$

So one has the Ricci scalar curvature,

$$\bar{R} = \bar{g}^{AB} \bar{R}_{AB} \quad (60)$$

$$= g^{00} \bar{R}_{00} + g^{11} \bar{R}_{11} + g^{22} \bar{R}_{22} + g^{33} \bar{R}_{33} \quad (61)$$

$$= 0. \quad (62)$$

From the fields  $\tilde{\mathcal{F}}_{\mu\nu}^m$  in (36) and (37), the components of the Ricci tensor, (56) ~ (59), can be recast into the form,

$$\bar{R}_{\mu\nu} = -\frac{1}{2} \bar{g}^{\alpha\beta} \gamma_{mn} \tilde{\mathcal{F}}_{\mu\alpha}^m \tilde{\mathcal{F}}_{\nu\beta}^n \quad (63)$$

Since the identity,  $\gamma_{mn} \tilde{\mathcal{F}}_{\mu\nu}^m \tilde{\mathcal{F}}^{\mu\nu n} = 0$ , holds, the right-hand side of the equation (63) can be identified as  $8\pi$  times the stress-energy tensor of the Yang-Mills field,

$$\bar{R}_{\mu\nu} = 8\pi \bar{T}_{\mu\nu}, \quad \bar{T}_{\mu\nu} = \frac{-1}{16\pi} \bar{g}^{\alpha\beta} \gamma_{mn} \tilde{\mathcal{F}}_{\mu\alpha}^m \tilde{\mathcal{F}}_{\nu\beta}^n. \quad (64)$$

Then the Einstein equation,  $\bar{R}_{AB} - \frac{1}{2} \bar{g}_{AB} \bar{R} = 8\pi \bar{T}_{AB}$ , is satisfied, where some components of  $\bar{T}_{AB}$  are zero,  $\bar{T}_{\mu m} = 0$  and  $\bar{T}_{mn} = 0$ .

### 3 Discussions

We have shown that the  $SO(3)$  dyon solution satisfies the Einstein equation in the seven-dimensional spacetimes. The stress-energy tensor of the Yang-Mills field in Einstein equation is derived from Ricci tensor automatically, not put by hand from outside. That the four-dimensional part of the KK dyonic metric is just the Schwarzschild black hole metric means the black hole in four-dimensional spacetimes and the dyonic gauge field are intimately related from the higher-dimensional spacetimes point of view. Since the dyon solution is spherically symmetric and the associated nonabelian symmetry group is  $SO(3)$ , many generalizations are possible. Applied to string theories maybe the most important direction for further investigations.

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